B.Sc. 3rd Semester (Honours) Examination, 2019-20

MATHEMATICS

Course ID : 32112 **Course Code : SHMTH-302/C-6**

Course Title: Group Theory-I

Time: 2 Hours **Full Marks: 40**

The figures in the right hand side margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

Notations and symbol have their usual meaning.

1. Answer *any five* questions: 2×5=10

(a) If G is an abelian group, then prove that $H = \{a \in G : a^2 = e\}$ is a subgroup of G.

(b) Give an example of an infinite group in which every element is of finite order.

(c) Find the order of the element $p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 7 & 1 & 4 & 2 & 6 & 9 & 10 & 3 & 5 & 8 \end{pmatrix}$ in S_{10}.

(d) Prove that the order of *An* is $\frac{n!}{2}$.

(e) Find the cyclic subgroup of $(\mathbb{Z}_{30}, +)$ generated by 25.

(f) Prove that there cannot be any epimorphism from the group $(\mathbb{Z}_{15}, +)$ onto the group $(\mathbb{Z}_8, +)$.

(g) Prove that any finite cyclic groups of same order are isomorphic.

(h) Prove that the groups $(\mathbb{Z}, +)$ and $(\mathbb{R}, +)$ are not isomorphic.

2. Answer *any four* questions: 5×4=20

(a) Let H and K be two subgroups of a group G. Prove that HK is a subgroup G if and only if $HK = KH$. 5

(b) (i) Solve the following equation in the Klein's 4-group $K - 4$:
$b^{-1}cx^2ac^3 = bac$, where $K_4 = \{e, a, b, c\}$.

(ii) Let $(G,*)$ be a finite group containing even number of elements. Prove that there exists at least one elements a, other than the identify e in G such that $a * a = e$ holds. 2+3=5

(c) (i) Prove that a finite semigroup satisfying the cancellation laws, forms a group.

(ii) Let G be a group. Show that $o(a) = o(b^{-1}ab) = o(bab^{-1})$. 3+2=5

(d) Let $n \geq 2$ and $\sigma \in S_n$ be a cycle. Them σ is a k-cycle iff order of σ is k. 5

(e) Let f be a homomorphism from the group G onto G_1. Then show that $\frac{G}{Kerf} \cong G_1$. 5

(f) (i) Show that the set of all rotations

$T(\theta): (x, y) \to (x', y')$ where $x' = x\cos\theta - y\sin\theta,\ y' = x\sin\theta + y\cos\theta$ forms a group with respect to multiplication.

(ii) Write down all the elements of the group U_{10}. 4+1=5

3. Answer *any one* question: 10×1=10

(a) (i) Define the Dihedral group D_4 of order 8. Find the centre of D_4.

(ii) Let $G = \left\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}:\ a, b, c \text{ are reals and } ac \neq 0\right\}$ be a group under usual matrix multiplication. Show that $N = \left\{\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}: c \text{ is real}\right\}$ is a normal subgroup of G.

(iii) Prove that every group of prime order is cyclic. (2+3)+3+2=10

(b) (i) Prove that the group $\frac{4\mathbb{Z}}{12\mathbb{Z}} \cong \mathbb{Z}_3$.

(ii) Let H be a subgroup of a group G such that $[G: H] = 2$. Then prove that H is a normal subgroup of G. Is the converse true? Justify your answer. 4+(3+3)=10

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